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PRECAUTIONARY SAVING AND RISK AVERSION
An Anticipated Utility Approach

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Yaari (1987) proved that within the anticipated utility framework, a risk averse decision maker will have precautionary saving regardless of the sign of the third derivative of his utility function. In this note we extend (a modification of) this result for an n-period model.

1. Introduction

One of the surprises that came with the development and applications of expected utility theory to economics was the realization that risk aversion alone does not give rise to precautionary saving. As modeled by Leland (1968), precautionary saving was interpreted as an increase in saving in the current period if future income has the same mean but becomes less certain. Leland found that existence of precautionary saving requires that the third derivative of the utility function is positive, or equivalently, that the marginal utility function is convex.

Leland's finding may be interpreted as an evidence that our intial understanding of precautionary saving was lacking. But it could also imply that the way risk is modeled in expected utility theory, solely through declining marginal utility, is unsatisfactory. The development of new theories of decision making under uncertainty by Machina (1982) and Quiggin (1982) provides a suitable framework to examine this possibility. This examination is the subject of our note.

The third derivative condition and its modification under the generalizations of expected utility theory can be illuminated by the following simple example. The consumer selects his first and second period consumption c_1 and c_2, given his income in both periods y_1 and y_2. y_1 is known with certainty, but y_2 is a random variable, being $\bar{y}_2 - \epsilon$ or $\bar{y}_2 + \epsilon$ with probability 1/2. For simplicity we assume a separable utility u, identical for both periods, zero interest rate, and no time preference. Under expected utility, the consumer solves

$$\max\ u(c_1) + \mathrm{E}[u(c_2)], \qquad \text{s.t. } c_2 = y_2 + y_1 - c_1.$$

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First-order conditions imply that

$$u'(c_1) = \mathrm{E}[u'(y_2 + y_1 - c_1)].$$

Compare now the case where y_2 is certain, i.e., $\epsilon = 0$, and the case where it is uncertain and $\epsilon > 0$. Using Jensen's Inequality, if u' is convex, then

$$u'(\bar{y}_2 + y_1 - c_1) < \tfrac{1}{2}u'(\bar{y}_2 - \epsilon + y_1 - c_1) + \tfrac{1}{2}u'(\bar{y}_2 + \epsilon + y_1 - c_1).$$

The convexity of u' thus implies an increase in the second period expected marginal utility and hence a decrease in first period consumption. In other words, convex marginal utility (i.e., a positive third derivative) induces precautionary saving. The sign of the third derivatives is of course independent of the sign of the second derivative, hence, precautionary saving is not implied by risk aversion.

As was mentioned before, one could dismiss the intuitive notion that risk aversion must lead to precautionary saving as a confusion between the effect on total utility and on marginal utility, but one could also question the notion of risk aversion implied by the use of the expected utility paradigm. Indeed, consider the consumer problem when expected utility is replaced by anticipated utility [Quiggin (1982)]. According to this theory, for $x_1 \leq \ldots \leq x_n$, the value of the lottery $(x_1, p_1; \ldots; x_n, p_n)$, yielding x_i dollars with probability p_i, $i = 1, \ldots, n$, is

$$u(x_1) + \sum_{i=2}^{n} [u(x_i) - u(x_{i-1})] f\left(\sum_{j=i}^{n} p_j\right)$$

$$= u(x_n) f(p_n) + \sum_{i=1}^{n-1} u(x_i)\left[f\left(\sum_{j=i}^{n} p_j\right) - f\left(\sum_{j=i+1}^{n} p_j\right)\right],$$

where the transformation function f satisfies $f(0) = 0$, $f(1) = 1$, $f' > 0$, and $f'' > 0$ for risk averse decision makers. The anticipated utility of the consumer in our model is

$$u(c_1) + u(y_1 - c_1 + \bar{y}_2 - \epsilon)[1 - f(\tfrac{1}{2})] + u(y_1 - c_1 + \bar{y}_2 + \epsilon) f(\tfrac{1}{2}),$$

and the first-order conditions imply that

$$u'(c_1) = u'(y_1 - c_1 + \bar{y}_2 - \epsilon)[1 - f(\tfrac{1}{2})] + u'(y_1 - c_1 + \bar{y}_2 + \epsilon) f(\tfrac{1}{2}).$$

For a risk averse consumer $f(1/2) < 1/2$, and as $u'' < 0$, the use of anticipated utility increases second period marginal utility and hence reduces first period consumption, as compared with expected utility. It is of course not always true that whenever ϵ goes up, the marginal anticipated utility of the second period goes up, as u' may be very concave. However, if ϵ is sufficiently small, then as u' is a differentiable function, higher ϵ necessarily means larger marginal utility in the second period, and hence precautionary saving.

This result, that a risk averse anticipated utility maximizer has precautionary saving was discovered by Yaari (1987). Yaari assumed a general utility function for consumption in the two periods, but noted that his approach cannot be extended to an n-period model. In this note we restrict the utility function to be additive, and show that a modification of Yaari's result can be formalized and proved within an n-period dynamic programming model (section 2). In section 3 we discuss these results, the limitation of our approach and conclude the paper.

2. The model

We consider the standard dynamic programming problem of a consumer whose planning horizon is $n+1$ periods. The utility function at period i is u_i, and c_i and $\tilde{y}_i$ are consumption and the random income at period i respectively. The decision in period i on consumption is made after this period's income has become known to be y_i. The problem is solved backwards from the last period (period n). Define recursively

$$v_n(y_n) \equiv u_n(y_n),$$

$$v_i(y_i) \equiv \max_{c_i \geq 0} u_i(c_i) + V\left(u_{i+1}^{-1}\left(v_{i+1}\left((y_i - c_i)r + \tilde{y}_{i+1}\right)\right)\right), \qquad i = 0, \ldots, n-1,$$

where V is the anticipated utility operator, specified in the introduction, and r is one plus the interest rate.

We now specialize the random variable $\tilde{y}_i$ as follows

$$\Pr(\tilde{y}_i = \bar{y}_i - \epsilon_i) = \Pr(\tilde{y}_i = \bar{y}_i + \epsilon_i) = \alpha_i,$$

$$\Pr(\tilde{y}_i = \bar{y}_i) = 1 - 2\alpha_i.$$

The consumer utility now becomes

$$v_i(y_i) = \max_{0 \leq c_i} u_i(c_i) + v_{i+1}\left((y_i - c_i)r + \bar{y}_{i+1} - \epsilon_{i+1}\right)\left[1 - f(1 - \alpha_{i+1})\right]$$

$$+ v_{i+1}\left((y_i - c_i)r + \bar{y}_{i+1}\right)\left[f(1 - \alpha_{i+1}) - f(\alpha_{i+1})\right] + v_{i+1}\left((y_i - c_i)r + \bar{y}_{i+1} + \epsilon_{i+1}\right)f(\alpha_{i+1}). \quad (1)$$

It is a standard result that if all the u_i functions are concave, then so are the v_i functions. Let $B \equiv v'_{i+1}((y_i - c_i)r + \bar{y}_{i+1} - \epsilon_{i+1})[1 - f(1 - \alpha_{i+1})] + v'_{i+1}((y_i - c_i)r + \bar{y}_{i+1})[f(1 - \alpha_{i+1}) - f(\alpha_{i+1})] + v'_{i+1}((y_i - c_i)r + \bar{y}_{i+1} + \epsilon_{i+1})f(\alpha_{i+1})$, and let C be the same as B with v''_{i+1} instead of v'_{i+1}.

The first-order condition for the maximum problem at the right-hand side of (1) is

$$u'_i(c_i) - rB = 0. \quad (2)$$

As u_i and v_{i+1} are concave, $u''_i(c_i) + r^2 C < 0$, and the second-order sufficient condition for maximum is satisfied.

We differentiate c_i implicitly with respect to y_i to obtain the standard normality result:

$$\frac{\mathrm{d}c_i}{\mathrm{d}y_i} = \frac{r^2 C}{u''_i(c_i) + r^2 C} > 0.$$

Note also that $\mathrm{d}c_i/\mathrm{d}y_i < 1$.

Consider now an increase in ϵ_{i+1} [this is of course a mean preserving increase in risk – see Rothschild and Stiglitz (1970)]. It follows from (2) that

$$\operatorname{sign}\frac{\mathrm{d}c_i}{\mathrm{d}\epsilon_{i+1}} = \operatorname{sign}\{ v''_{i+1}((y_i - c_i)r + \bar{y}_{i+1} - \epsilon_{i+1})[1 - f(1-\alpha_{i+1})]$$

$$- v''_{i+1}((y_i - c_i)r + \bar{y}_{i+1} + \epsilon_{i+1}) f(\alpha_{i+1})\}.$$

Since v_{i+1} is concave, both expressions are negative. If f is linear, i.e., if the anticipated utility functional is reduced to the expected utility functional, then the sign of $\mathrm{d}c_i/\mathrm{d}\epsilon_{i+1}$ is opposite to the sign of u'''_{i+1}. However, if f is convex (as we already assumed in accordance with risk aversion), then $1 - f(1-\alpha_{i+1}) > f(\alpha_{i+1})$ and for a sufficiently small ϵ_{i+1}, $\mathrm{d}c_i/d\epsilon_{i+1} < 0$. The neighborhood for ϵ_{i+1} can be determined independently of α_{i+1}, since the difference $1 - f(1-\alpha_{i+1}) - f(\alpha_{i+1})$ is bounded from below by $1 - 2f(1/2)$ which is strictly positive.

We have thus proved the following proposition.

Proposition. *Let a consumer have concave utilities* u_i, *income* $\tilde{y}_i$, *where* $\Pr(\tilde{y}_i = \bar{y}_i - \epsilon_i) = \Pr(\tilde{y}_i = \bar{y}_i + \epsilon_i) = \alpha_i$, $\Pr(\tilde{y}_i = \bar{y}_i) = 1 - 2\alpha_i$, $i = 0, \ldots, n$, $\alpha_0 = 0$, *and a convex transformation function f. There exists* $\epsilon > 0$ *such that if* $\epsilon_{i+1} < \epsilon$, *then a small increase in* ϵ_{i+1} *will result in a decline in* c_i. *This* ϵ *is determined independently of* α_{i+1}.

This proposition can be easily extended to cover the effect of an increase in ϵ_i on consumption in all periods. By the proposition, for all values of y_{i-1}, $\mathrm{d}c_{i-1}/\mathrm{d}\epsilon_i < 0$. By the Envelope Theorem, $v'_{i-1}(y_{i-1}) = u'_{i-1}(c_{i-1}(y_{i-1}))$. As c_{i-1} declines, $u'_{i-1}(c_{i-1})$ increases and so does v'_{i-1}. From the first-order conditions and the concavity of v_{i-1} it follows that c_{i-2} decreases. By the same logic, $c_{i-3}, c_{i-4}, \ldots, c_0$ all decline when ϵ_i icreases.

The effect of the same increase in ϵ_i on $c_i, \ldots, c_n$ is naturally the opposite. Since more is saved in former periods, y_i increases and so do $c_i, c_{i+1}, \ldots, c_n$, due to the fact that $0 < \mathrm{d}c_i/\mathrm{d}y_i < 1$. This discussion is summarized in the following theorem:

Theorem. *Let a consumer have concave utility functions* $u_0, \ldots, u_n$, *a convex transformation function f, and income profiles* $\tilde{y}_0, \ldots, \tilde{y}_n$ *as in the proposition. There exists* $\epsilon > 0$ *such that if* $\epsilon_i < \epsilon$, *an increase in* ϵ_i *will result in a decrease in* $c_0, \ldots, c_{i-1}$ *and in an increase in* $c_i, \ldots, c_n$.

3. Summary and conclusion

In this paper we extend Yaari's (1987) results, that the interpretation of risk aversion via anticipated utility theory results in precautionary saving as a reaction to increase in uncertainty, to an n-period model. The notion of increase in uncertainty used by us was essentially the substitution of a random variable for its mean, as was used by Arrow (1974). To obtain this result we had to assume that the random variable is bounded in a neighborhood around its mean. This neighborhood is independent of the structure of the random variable, and if the third derivative of the utility function is bounded, it can also be made independent of the level of average income in all periods. Unfortunately, our results cannot be extended to the more general notion of increase in uncertainty, the mean preserving increase in risk [Rothschild and Stiglitz (1970)]. To see this, consider the random variable $\tilde{y}_i$ as defined above and increase α_i, which is a mean preserving spread. By the same approach as used in section 2 it follows that

$$\begin{aligned}\operatorname{sign}\frac{\mathrm{d}c_i}{\mathrm{d}\alpha_{i+1}} &= \operatorname{sign}\{f'(1-\alpha_{i+1})[v'_{i+1}((y_i-c_i)r+\bar{y}_{i+1}-\epsilon_{i+1})-v'_{i+1}((y_i-c_i)r+\bar{y}_{i+1})] \\ &\quad -f'(\alpha_{i+1})[v'_{i+1}((y_i-c_i)r+\bar{y}_{i+1})-v'_{i+1}((y_i-c_i)r+\bar{y}_{i+1}+\epsilon_{i+1})]\} \\ &= \operatorname{sign}\{f'(\alpha_{i+1})v''_{i+1}((y_i-c_i)r+\tilde{y}_{i+1}+\theta_1\epsilon_{i+1}) \\ &\quad -f'(l-\alpha_{i+1})v''_{i+1}((y_i-c_i)r+\tilde{y}_{i+1}-\theta_2\epsilon_{i+1})],\end{aligned}$$

where $0<\theta_1,\theta_2<1$. It is still true that for every α_{i+1} there is ϵ such that if $\epsilon_{i+1}<\epsilon$, $\mathrm{d}c_i/\mathrm{d}\alpha_{i+1}<0$, but this ϵ depends on α_{i+1}, moreover, as α_{i+1} approaches 1/2, ϵ approaches 0, and the sign of $\mathrm{d}c_i/\mathrm{d}\alpha_{i+1}$ depends on the third derivative of u.

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